

Trigonometric Identities and Equations

TRIGONOMETRIC IDENTITIES AND EQUATIONS

Section - 1

1. INTRODUCTION

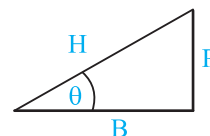
1.1 Trigonometric Ratios for acute angles:

For an acute angle θ , *Trigonometric Ratios* (T-ratios) can be defined using a right angled triangle with angles θ , $90^\circ - \theta$, 90° .

$$\sin\theta = \frac{P}{H}$$

$$\cos\theta = \frac{B}{H}$$

$$\tan\theta = \frac{P}{B}$$



$\operatorname{cosec}\theta$, $\sec\theta$, $\cot\theta$ are reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively.

Some standard identities for above trigonometric ratios are :

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

For acute angles, all T-ratios are positive.

1.2. ANGLE

An angle is the amount of rotation of a revolving line with respect to a fixed line:

There are three systems of measuring an angle :

- (i) Sexagesimal system or English system
- (ii) Centesimal or french system
- (iii) Circular system

(i) Sexagesimal System :

In this system a right angle is divided into 90 equal parts, called degrees. The symbol 1° is used to denote one degree. Thus, one degree is one-ninetieth part of right angle. Each degree is divided into 60 equal parts, called minutes and one minute is divided into 60 equal parts, called seconds. The symbol $1'$ and $1''$ are used to denote one minute and one second, respectively.

Thus, 1 right angle = 90 degrees ($= 90^\circ$)

$$1^\circ = 60 \text{ minutes } (= 60')$$

$$1' = 60 \text{ seconds } (= 60'')$$

(ii) Centesimal System :

In this system a right angle is divided into 100 equal parts, called grades; each grade is subdivided into 100 minutes each minute is divided into 100 seconds.

The symbol 1^g , $1'$ and $1''$ are used to denote a grade, a minute and a second respectively.

Thus, 1 right angle = 100 grades (= 100^g)

1 grade = 100 minutes (= 100')

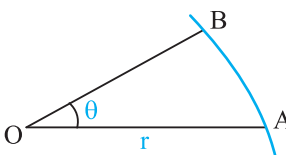
1 minute = 100 seconds (= 100'')

(iii) Circular System :

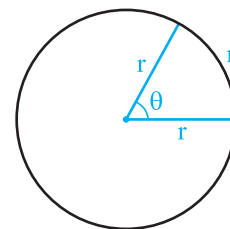
In this system the unit of measurement is radian as defined below.

Radians : One radian, written as 1^c, is the measure of an angle subtended at the centre of circle by an arc of length equal to radius of the circle.

Consider an arc AB of a circle of radius r subtending an angle θ at its centre. The ratio between length of arc AB and the radius of circle represents the measure of angle θ in the radians. i.e.

$$\theta \text{ (in radians)} = \frac{\text{arc } AB}{r}$$


One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of circle.



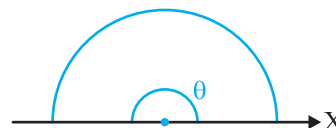
Relation between degree and radians :

Let θ be the angle subtended at centre by a semicircle = 180°.

From figure, $\theta = 180^\circ$.

In radians: $\theta = \frac{\text{arc}}{\text{radius}} = \frac{\pi r}{r} = \pi$

$$\Rightarrow 180^\circ \text{ degrees} = \pi \text{ radian}$$



Radian is an important unit for measuring angles. So the following conversions must be remembered

Degree	0	30	45	60	90	120	135	150	180	270
Radian	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	$3\pi/2$

Note : θ radian is written as θ^c or can be written simply as θ . When the unit of angle is not mentioned, it must be taken as radians.

Measuring of angle in various quadrants :

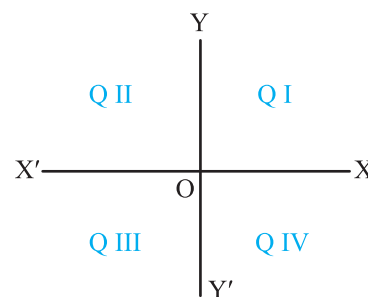
The perpendicular lines XOX' and YOY' divide the plane in four parts :

Q I : First Quadrant (all points have + X and + Y)

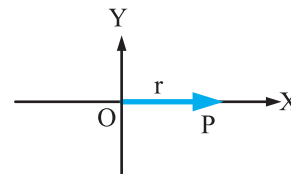
Q II : Second Quadrant (all points have - X and + Y)

Q III : Third Quadrant (all points have - X and - Y)

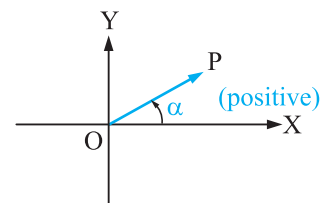
Q IV : Fourth Quadrant (all points have + X and - Y)



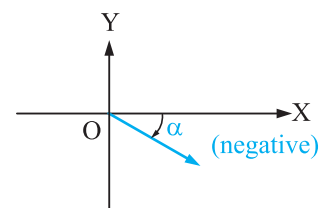
- ▶ The angles in trigonometry can be positive or negative and can have any magnitude. Every angle is represented by one position of a revolving ray OP of length r . The starting position for ray OP is taken along $+X$ axis.



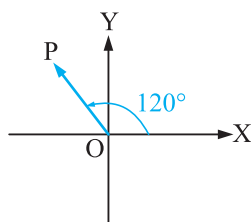
- ▶ If an angle α (alpha) is positive, OP rotates through angle α in anticlockwise direction.



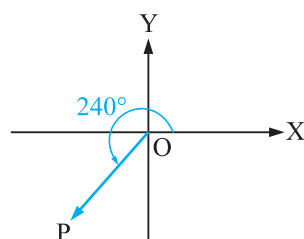
- ▶ If an angle α is negative, OP rotates through angle α in clockwise direction.



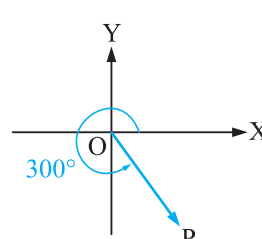
- ▶ An angle can lie in any of four quadrants according to the position of revolving ray for the angle.



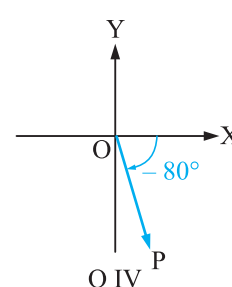
Q II



Q III

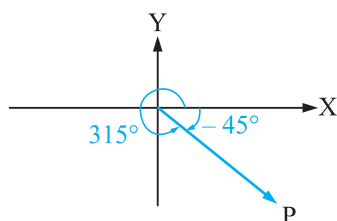


Q IV

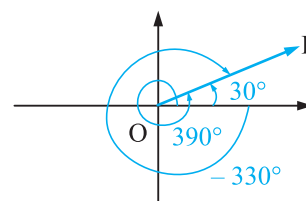


Q IV

- ▶ Two or more angles may correspond to same position of revolving ray OP .



This position of OP represents -45° and $+315^\circ$



This position of OP represents $+30^\circ$, $+390^\circ$ and -330°

- ▶ The position of revolving ray for angles 0° , 30° , 45° , 60° , 90° , 120° , 135° , 180° , 270° , 360° must be remembered.

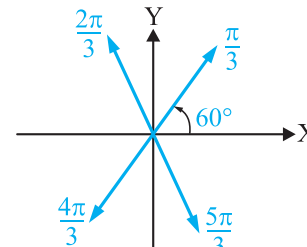
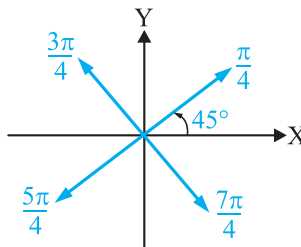
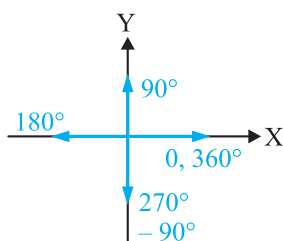


Illustration - 1 In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.

- (A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{5\pi}{3}$ (D) $\frac{30\pi}{3}$

SOLUTION : (B)

Let arc $AB = S$. It is given that $OA = 20$ cm and chord $AB = 20$ cm.

Therefore, $\triangle OAB$ is an equilateral triangle.

$$\Rightarrow \angle AOB = 60^\circ = \left(60 \times \frac{\pi}{180}\right) = \left(\frac{\pi}{3}\right)^c \quad (\text{in radians})$$

$$\text{We know that } \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{3} = \frac{S}{20} \Rightarrow S = \frac{20\pi}{3} \text{ cm}$$

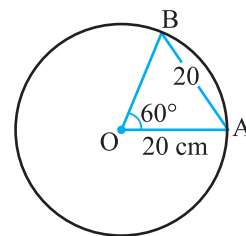


Illustration - 2 Find the radian measure corresponding to $-37^\circ 30'$.

- (A) $\frac{-5\pi}{24}$ (B) $\frac{\pi}{24}$ (C) $\frac{5\pi}{24}$ (D) $\frac{-7\pi}{24}$

SOLUTION : (A)

In such kind of problems first of all we convert minute into degree and then degree into radians

Therefore,

$$60' = 1^\circ$$

$$\Rightarrow 30' = \left(\frac{1}{2}\right)^\circ$$

$$\Rightarrow -37^\circ 30' = -37\frac{1}{2}^\circ = -\frac{75^\circ}{2}$$

$$\text{Now, } 360^\circ = 2\pi \text{ radians}$$

$$\Rightarrow \frac{-75^\circ}{2} = -\frac{2\pi}{360} \times \frac{75}{2} \text{ radians} = \frac{-5\pi}{24} \text{ radians} \quad \left[\text{using } 1^\circ = \frac{\pi}{180} \text{ radians}\right]$$

This (–) sign indicates that measure of angle is the clockwise direction.

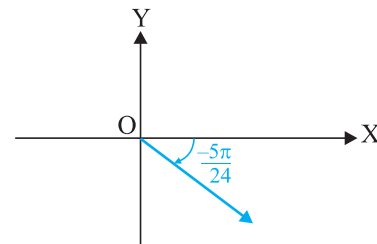


Illustration - 3 If the angles of a triangle are in the ratio 3 : 4 : 5, find the smallest angle in degrees and the greatest angle in radians.

- (A) $50^\circ, \frac{\pi}{12}$ (B) $70^\circ, \frac{\pi}{12}$ (C) $75^\circ, \frac{5\pi}{12}$ (D) $85^\circ, \frac{\pi}{12}$

SOLUTION : (C)

Let the three angles be $3x$, $4x$ and $5x$ degrees,

In triangle,

$$3x + 4x + 5x = 180$$

$$\Rightarrow 12x = 180 \Rightarrow x = 15$$

$$\Rightarrow \text{The smallest angle} = 3x \text{ degrees}$$

$$= 3 \times 15 \text{ degree} = 45^\circ$$

and the greatest angle = $5x$ degree

$$= 5 \times 15 \text{ degree} = 75^\circ$$

$$= \left(75 \times \frac{\pi}{180}\right) \text{ radians} = \frac{5\pi}{12} \text{ radians}$$